6 Hilbert Spaces

6.1 The Fréchet-Riesz Representation Theorem

Definition 6.1. Let X be an inner product space. A pair of vectors $x, y \in X$ is called *orthogonal* iff $\langle x, y\rangle = 0$. We write $x \perp y$. A pair of subsets $A, B \subseteq X$ is called *orthogonal* iff $x \perp y$ for all $x \in A$ and $y \in B$. Moreover, if $A \subseteq X$ is some subset we define its *orthogonal complement* to be

$$A^\perp := \{y \in X : x \perp y \ \forall x \in A\}.$$

<u>**Exercise**</u> **30.** Let X be an inner product space.

1. Let $x, y \in X$. If $x \perp y$ then $\|x\|^2 + \|y\|^2 = \|x + y\|^2$.
2. Let $A \subseteq X$ be a subset. Then $A^\perp$ is a closed subspace of X.
3. $A \subseteq (A^\perp)^\perp$.
4. $A^\perp = \overline{(\operatorname{span} A)}^\perp$.
5. $A \cap A^\perp \subseteq \{0\}$.

Proposition 6.2. *Let H be a Hilbert space, $F \subseteq H$ a closed and convex subset and $x \in H$. Then, there exists a unique element $\tilde{x} \in F$ such that*

$$\|\tilde{x} - x\| = \inf_{y\in F} \|y - x\|.$$

Proof. Define $a := \inf_{y\in F} \|y-x\|$. Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence in F such that $\lim_{n\to\infty} \|y_n - x\| = a$. Let $\epsilon > 0$ and choose $n_0 \in \mathbb{N}$ such that $\|y_n - x\|^2 \le a^2 + \epsilon$ for all $n \ge n_0$. Now let $n, m \ge n_0$. Then, using the parallelogram equality of Theorem 2.39 we find

$$\begin{aligned}
\|y_n - y_m\|^2 &= 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - \|y_n + y_m - 2x\|^2\\
&= 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4\left\|\frac{y_n + y_m}{2} - x\right\|^2\\
&\le 2(a^2 + \epsilon) + 2(a^2 + \epsilon) - 4a^2 = 4\epsilon
\end{aligned}$$

This shows that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence which must converge to some vector $\tilde{x} \in F$ with the desired properties since F is complete.

It remains to show that $\tilde{x}$ is unique. Suppose $\tilde{x}, \tilde{x}' \in F$ both satisfy the condition. Then, by a similar use of the parallelogram equation as above,

$$\|\tilde{x}-\tilde{x}'\|^2 = 2\|\tilde{x}-x\|^2+2\|\tilde{x}'-x\|^2-4\left\|\frac{\tilde{x}+\tilde{x}'}{2} - x\right\|^2 \le 2a^2+2a^2-4a^2 = 0.$$

That is, $\tilde{x}' = \tilde{x}$, completing the proof. $\square$

Lemma 6.3. *Let H be a Hilbert space, $F \subseteq H$ a closed and convex subset, $x \in H$ and $\tilde{x} \in H$. Then, the following are equivalent:*

1. $\|\tilde{x} - x\| = \inf_{y\in F} \|y - x\|$
2. $\Re\langle \tilde{x} - y, \tilde{x} - x\rangle \leq 0\ \forall y \in F$

Proof. Suppose 2. holds. Then, for any $y \in F$ we have

$$\begin{aligned}\|y - x\|^2 &= \|(y - \tilde{x}) + (\tilde{x} - x)\|^2 \\ &= \|y - \tilde{x}\|^2 + 2\Re\langle y - \tilde{x}, \tilde{x} - x\rangle + \|\tilde{x} - x\|^2 \geq \|\tilde{x} - x\|^2.\end{aligned}$$

Conversely, suppose 1. holds. Fix $y \in F$ and consider the continuous map $[0, 1] \to F$ given by $t \mapsto y_t := (1 - t)\tilde{x} + ty$. Then,

$$\begin{aligned}\|\tilde{x} - x\|^2 \leq \|y_t - x\|^2 &= \|t(y - \tilde{x}) + (\tilde{x} - x)\|^2 \\ &= t^2\|y - \tilde{x}\|^2 + 2t\Re\langle y - \tilde{x}, \tilde{x} - x\rangle + \|\tilde{x} - x\|^2.\end{aligned}$$

Subtracting $\|\tilde{x} - x\|^2$ and dividing for $t \in (0, 1]$ by t leads to,

$$\frac{1}{2}t\|y - \tilde{x}\|^2 \geq \Re\langle \tilde{x} - y, \tilde{x} - x\rangle.$$

This implies 2. □

Lemma 6.4. *Let H be a Hilbert space, $F \subseteq H$ a closed subspace, $x \in H$ and $\tilde{x} \in F$. Then, the following are equivalent:*

1. $\|\tilde{x} - x\| = \inf_{y\in F} \|y - x\|$
2. $\langle y, \tilde{x} - x\rangle = 0\ \forall y \in F$

Proof. **Exercise.** □

Proposition 6.5. *Let H be a Hilbert space, $F \subseteq H$ a closed proper subspace. Then, $F^{\perp} \neq \{0\}$.*

Proof. Since F is proper, there exists $x \in H \setminus F$. By Proposition 6.2 there exists an element $\tilde{x} \in F$ such that $\|\tilde{x} - x\| = \inf_{y\in F} \|y - x\|$. By Lemma 6.4, $\langle y, \tilde{x} - x\rangle = 0$ for all $y \in F$. That is, $\tilde{x} - x \in F^{\perp}$. □

Theorem 6.6 (Fréchet-Riesz Representation Theorem)**.** *Let H be a Hilbert space. Then, the map $\Phi : H \to H^*$ given by $(\Phi(x))(y) := \langle y, x\rangle$ for all $x, y \in H$ is anti-linear, bijective and isometric.*

Proof. The anti-linearity of Φ follows from the properties of the scalar product. Observe that for all $x \in H$, $\|\Phi(x)\| = \sup_{\|y\|=1} |\langle y, x\rangle| \le \|x\|$ because of the Schwarz inequality (Theorem 2.35). On the other hand, $(\Phi(x))(x/\|x\|) = \|x\|$ for all $x \in H \setminus \{0\}$. Hence, $\|\Phi(x)\| = \|x\|$ for all $x \in H$, i.e., Φ is isometric. It remains to show that Φ is surjective. Let $f \in H^* \setminus \{0\}$. Then $\ker f$ is a closed proper subspace of H and by Proposition 6.5 there exists a vector $v \in (\ker f)^\perp \setminus \{0\}$. Observe that for all $x \in H$,

$$x - \frac{f(x)}{f(v)} v \in \ker f.$$

Hence,

$$\langle x, v\rangle = \left\langle x - \frac{f(x)}{f(v)} v + \frac{f(x)}{f(v)} v, v \right\rangle = \frac{f(x)}{f(v)} \langle v, v\rangle$$

In particular, setting $w := \overline{f(v)}/(\|v\|^2)$ we see that $\Phi(w) = f$. □

Corollary 6.7. *Let H be a Hilbert space. Then, H^* is also a Hilbert space. Moreover, H is reflexive, i.e., H^{**} is naturally isomorphic to H.*

Proof. By Theorem 6.6 the spaces H and H' are isometric. This implies in particular, that H' is complete that its norm satisfies the parallelogram equality, i.e., that it is a Hilbert space. Indeed, it is easily verified that the inner product is given by

$$\langle \Phi(x), \Phi(y)\rangle_{H'} = \langle y, x\rangle_H \quad \forall x, y \in H.$$

Consider the canonical linear map $i_H : H \to H^{**}$. It is easily verified that $i_H = \Psi \circ \Phi$, where $\Psi : H^* \to H^{**}$ is the corresponding map of Theorem 6.6. Thus, i_H is a linear bijective isometry, i.e., an isomorphism of Hilbert spaces. □

6.2 Orthogonal Projectors

Theorem 6.8. *Let H be a Hilbert space and $F \subseteq H$ a closed subspace such that $F \neq \{0\}$. Then, there exists a unique operator $P_F \in \mathrm{CL}(H, H)$ with the following properties:*

1. $P_F|_F = \mathbf{1}_F$.
2. $\ker P_F = F^\perp$.

Moreover, P_F also has the following properties:

3. *$P_F(H) = F$.*

4. *$P_F \circ P_F = P_F$.*

5. *$\|P_F\| = 1$.*

6. *Given $x \in H$, $P_F(x)$ is the unique element of F such that $\|P_F(x)-x\| = \inf_{y\in F} \|y - x\|$.*

7. *Given $x \in H$, $P_F(x)$ is the unique element of F such that $x - P(x) \in F^\perp$.*

Proof. We define P_F to be the map $x \mapsto \tilde{x}$ given by Proposition 6.2. Then, clearly $P_F(H) = F$ and $P_F(x) = x$ if $x \in F$ and thus $P_F \circ P_F = P_F$. By Lemma 6.4 we have $P_F(x) - x \in F^\perp$ for all $x \in H$. Since $F^\perp$ is a subspace we have

$$(\lambda_1 P_F(x_1) - \lambda 1 x_1) + (\lambda_2 P_F(x_2) - \lambda_2 x_2) \in F^\perp$$

for $x_1, x_2 \in H$ and $\lambda_1, \lambda_2 \in \mathbb{K}$ arbitrary. Rewriting this we get,

$$(\lambda_1 P_F(x_1) - \lambda_2 P_F(x_2)) - (\lambda_1 x_1 + \lambda_2 x_2) \in F^\perp.$$

But Lemma 6.4 also implies that if given $x \in H$ we have $z - x \in F^\perp$ for some $z \in F$, then $z = P_F(x)$. Thus,

$$\lambda_1 P_F(x_1) - \lambda_2 P_F(x_2) = P_F(\lambda_1 x_1 + \lambda_2 x_2).$$

That is, P_F is linear. Using again that $x - P_F(x) \in F^\perp$ we have $x - P_F(x) \perp P_F(x)$ and hence the Pythagoras equality (Exercise 30.1)

$$\|x - P_F(x)\|^2 + \|P_F(x)\|^2 = \|x\|^2 \quad \forall x \in H.$$

This implies $\|P_F(x)\| \leq \|x\|$ for all $x \in H$. In particular, P_F is continuous. On the other hand $\|P_F(x)\| = \|x\|$ if $x \in F$. Therefore, $\|P_F\| = 1$. Now suppose $x \in \ker P_F$. Then, $\langle y, x\rangle = -\langle y, P_F(x) - x\rangle = 0$ for all $y \in F$ and hence $x \in F^\perp$. That is, $\ker P_F \subseteq F^\perp$. Conversely, suppose now $x \in F^\perp$. Then, $\langle y, P_F(x)\rangle = \langle y, P_F(x)-x\rangle = 0$ for all $y \in F$. Thus, $P_F(x) \in F^\perp$. But we know already that $P_F(x) \in F$. Since, $F \cap F^\perp = \{0\}$ we get $P_F(x) = 0$, i.e., $x \in \ker P_F$. Then, $F^\perp \subseteq \ker P_F$. Thus, $\ker P_F = F^\perp$. This concludes the proof the the existence of P_F with properties 1, 2, 3, 4, 5, 6 and 7.

Suppose now there is another operator $Q_F \in \mathrm{CL}(H, H)$ which also has the properties 1 and 2. We proceed to show that $Q_F = P_F$. To this end, consider the operator $\mathbf{1} - P_F \in \mathrm{CL}(H, H)$. We then have $(\mathbf{1} - P_F)(x) = x - P_F(x) \in F^\perp$ for all $x \in H$. Thus, $(\mathbf{1}-P_F)(H) \subseteq F^\perp$. Since $P_F+(\mathbf{1}-P_F) = \mathbf{1}$

and $P_F(H) = F$ we must have $H = F + F^\perp$. Thus, given $x \in H$ there are $x_1 \in F$ and $x_2 \in F^\perp$ such that $x = x_1 + x_2$. By property 1 we have $P_F(x_1) = Q_F(x_1)$ and by property 2 we have $P_F(x_2) = Q_F(x_2)$. Hence, $P_F(x) = Q_F(x)$. □

Definition 6.9. Given a Hilbert space H and a closed subspace F, the operator $P_F \in \mathrm{CL}(H, H)$ constructed in Theorem 6.8 is called the *orthogonal projector* onto the subspace F.

Corollary 6.10. *Let H be a Hilbert space and F a closed subspace. Let P_F be the associated orthogonal projector. Then $\mathbf{1} - P_F$ is the orthogonal projector onto $F^\perp$. That is, $P_{F^\perp} = \mathbf{1} - P_F$.*

Proof. Let $x \in F^\perp$. Then, $(\mathbf{1} - P_F)(x) = x$ since $\ker P_F = F^\perp$ by Theorem 6.8.1. That is, $(\mathbf{1} - P_F)|_{F^\perp} = \mathbf{1}_{F^\perp}$. On the other hand, suppose $(\mathbf{1}-P_F)(x) = 0$. By Theorem 6.8.1. and 3. this is equivalent to $x \in F$. That is, $\ker(\mathbf{1} - P_F) = F$. Applying Theorem 6.8 to $F^\perp$ yields the conclusion $P_{F^\perp} = \mathbf{1} - P_F$ due to the uniqueness of $P_{F^\perp}$. □

Corollary 6.11. *Let H be a Hilbert space and F a closed subspace. Then, $F = (F^\perp)^\perp$.*

Proof. **Exercise.** □

Definition 6.12. Let H_1 and H_2 be inner product spaces. Then, $H_1 \oplus_2 H_2$ denotes the direct sum as a vector space with the inner product

$$\langle x_1 + x_2, y_1 + y_2 \rangle := \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle \quad \forall x_1, x_2 \in H_1, \forall y_1, y_2 \in H_2.$$

Proposition 6.13. *Let H_1 and H_2 be inner product spaces. Then, the topology of $H_1 \oplus_2 H_2$ agrees with the topology of the direct sum of H_1 and H_2 as tvs. That is, it agrees with the product topology of $H_1 \times H_2$. In particular, if H_1 and H_2 are complete, then $H_1 \oplus_2 H_2$ is complete.*

Proof. **Exercise.** □

Corollary 6.14. *Let H be a Hilbert space and F a closed subspace. Then, $H = F \oplus_2 F^\perp$.*

Proof. **Exercise.** □

6.3 Orthonormal Bases

Definition 6.15. Let H be a Hilbert space and $S \subseteq H$ a subset such that $\|s\| = 1$ for all $s \in S$ and such that $\langle s, t\rangle \neq 0$ for $s, t \in S$ implies $s = t$. Then, S is called an *orthonormal system* in H. Suppose furthermore that S is maximal, i.e., that for any orthonormal system T in H such that $S \subseteq T$ we have $S = T$. Then, S is called an *orthonormal basis* of H.

Proposition 6.16. *Let H be a Hilbert space and S an orthonormal system in H. Then, S is linearly independent.*

Proof. **Exercise.** □

Proposition 6.17 (Gram-Schmidt)**.** *Let H be a Hilbert space and $\{x_n\}_{n\in I}$ be a linearly independent subset, indexed by the countable set I. Then, there exists an orthogonal system $\{s_n\}_{n\in I}$, also indexed by I and such that* $\operatorname{span}\{s_n : n \in I\} = \operatorname{span}\{x_n : n \in I\}$.

Proof. If I is finite we identify it with $\{1, \ldots, m\}$ for some $m \in \mathbb{N}$. Otherwise we identify I with $\mathbb{N}$. We construct the set $\{s_n\}_{n\in I}$ iteratively. Set $s_1 := x_1/\|x_1\|$. (Note that $x_n \neq 0$ for any $n \in I$ be the assumption of linear independence.) We now suppose that $\{s_1, \ldots, s_k\}$ is an orthonormal system and that $\operatorname{span}\{s_1, \ldots, s_k\} = \operatorname{span}\{x_1, \ldots, x_k\}$. Set $X_k := \operatorname{span}\{x_1, \ldots, x_k\}$. By linear independence $y_{k+1} := x_{k+1} - P_{X_k}(x_{k+1}) \neq 0$. Set $s_{k+1} := y_{k+1}/\|y_{k+1}\|$. Clearly, $s_{k+1} \perp X_K$, i.e., $\{s_1, \ldots, s_{k+1}\}$ is an orthonormal system. Moreover, $\operatorname{span}\{s_1, \ldots, s_{k+1}\} = \operatorname{span}\{x_1, \ldots, x_{k+1}\}$. If I is finite this process terminates, leading to the desired result. If I is infinite, it is clear that this process leads to $\operatorname{span}\{s_n : n \in \mathbb{N}\} = \operatorname{span}\{x_n : n \in \mathbb{N}\}$. □

Proposition 6.18 (Bessel's inequality)**.** *Let H be a Hilbert space, $m \in \mathbb{N}$ and $\{s_1, \ldots, s_m\}$ an orthonormal system in H. Then, for all $x \in H$,*

$$\sum_{n=1}^{m} |\langle x, s_n\rangle|^2 \leq \|x\|^2$$

Proof. Define $y := x - \sum_{n=1}^{m} \langle x, s_n\rangle s_n$. Then, $y \perp s_n$ for all $n \in \{1, \ldots, m\}$. Thus, applying Pythagoras we obtain

$$\|x\|^2 = \|y\|^2 + \left\| \sum_{n=1}^{m} \langle x, s_n\rangle s_n \right\|^2 = \|y\|^2 + \sum_{n=1}^{m} |\langle x, s_n\rangle|^2.$$

This implies the inequality. □

Lemma 6.19. *Let H be a Hilbert space, $S \subset H$ an orthonormal system and $x \in H$. Then, $S_x := \{s \in S : \langle x, s\rangle \neq 0\}$ is countable.*

Proof. **Exercise.**Hint: Use Bessel's Inequality (Proposition 6.18). □

Proposition 6.20 (Generalized Bessel's inequality)**.** *Let H be a Hilbert space, $S \subseteq H$ an orthonormal system and $x \in H$. Then*

$$\sum_{s\in S} |\langle x, s\rangle|^2 \leq \|x\|^2.$$

Proof. By Lemma 6.19, the subset $S_x := \{s \in S : \langle x, s\rangle \neq 0\}$ is countable. If S_x is finite we are done due to Proposition 6.18. Otherwise let $\alpha : \mathbb{N} \to S_x$ be a bijection. Then, by Proposition 6.18

$$\sum_{n=1}^{m} |\langle x, s_{\alpha(n)}\rangle|^2 \leq \|x\|^2$$

For any $m \in \mathbb{N}$. Thus, we may take the limit $m \to \infty$ on the left hand side, showing that the series converges absolutely and satisfies the inequality. □

Definition 6.21. Let X be a tvs and $\{x_i\}_{i\in I}$ an indexed set of elements of X. We say that the series $\sum_{i\in I} x_i$ *converges unconditionally* to $x \in X$ iff $I_0 := \{i \in I : x_i \neq 0\}$ is countable and for any bijection $\alpha : \mathbb{N} \to I$ the sum $\sum_{n=1}^{\infty} x_{\alpha(n)}$ converges to x.

Proposition 6.22. *Let H be a Hilbert space and $S \subset H$ an orthonormal system. Then, $P(x) := \sum_{s\in S}\langle x, s\rangle s$ converges unconditionally. Moreover, $P : x \mapsto P(x)$ defines an orthogonal projector onto $\overline{\operatorname{span} S}$.*

Proof. Fix $x \in H$. We proceed to show that $\sum_{s\in S}\langle x, s\rangle s$ converges unconditionally. The set S can be replaced by the set $S_x := \{s \in S : \langle x, s\rangle \neq 0\}$, which is countable due to Lemma 6.19. If S_x is even finite we are done. Otherwise, let $\alpha : \mathbb{N} \to S_x$ be a bijection. Then, given $\epsilon > 0$ by Proposition 6.20 there is $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0+1}^{\infty} |\langle x, s_{\alpha(n)}\rangle|^2 < \epsilon^2.$$

For $m > k \geq n_0$ this implies using Pythagoras,

$$\begin{aligned} \left\| \sum_{n=1}^{m}\langle x, s_{\alpha(n)}\rangle s_{\alpha(n)} - \sum_{n=1}^{k}\langle x, s_{\alpha(n)}\rangle s_{\alpha(n)} \right\|^2 &= \left\| \sum_{n=k+1}^{m} \langle x, s_{\alpha(n)}\rangle s_{\alpha(n)} \right\|^2 \\ &= \sum_{n=k+1}^{m} |\langle x, s_{\alpha(n)}\rangle|^2 < \epsilon^2. \end{aligned}$$

So the sequence $\{\sum_{n=1}^{m} \langle x, s_{\alpha(n)}\rangle s_{\alpha(n)}\}_{m\in\mathbb{N}}$ is Cauchy and must converge to some element $y_\alpha \in H$ since H is complete. Now let $\beta : \mathbb{N} \to S_x$ be another bijection. Then, $\sum_{n=1}^{\infty} \langle x, s_{\beta(n)}\rangle s_{\beta(n)} = y_\beta$ for some $y_\beta \in H$. We need to show that $y_\beta = y_\alpha$. Let $m_0 \in \mathbb{N}$ such that $\{\alpha(n) : n \le n_0\} \subseteq \{\beta(n) : n \le m_0\}$. Then, for $m \ge m_0$ we have (again using Pythagoras)

$$\left\| \sum_{n=1}^{m} \langle x, s_{\beta(n)}\rangle s_{\beta(n)} - \sum_{n=1}^{n_0} \langle x, s_{\alpha(n)}\rangle s_{\alpha(n)} \right\|^2 \le \sum_{n=n_0+1}^{\infty} |\langle x, s_{\alpha(n)}\rangle|^2 < \epsilon^2.$$

Taking the limit $m \to \infty$ we find

$$\left\| y_\beta - \sum_{n=1}^{n_0} \langle x, s_{\alpha(n)}\rangle s_{\alpha(n)} \right\| < \epsilon.$$

But on the other hand we have,

$$\left\| y_\alpha - \sum_{n=1}^{n_0} \langle x, s_{\alpha(n)}\rangle s_{\alpha(n)} \right\| < \epsilon.$$

Thus, $\|y_\beta - y_\alpha\| < 2\epsilon$. Since ϵ was arbitrary this shows $y_\beta = y_\alpha$ proving the unconditional convergence.

It is now clear that $x \mapsto P(x)$ yields a well defined map $P : H \to H$. From the definition it is also clear that $P(H) \subseteq \overline{\operatorname{span} S}$. Let $s \in S$. Then,

$$\langle x - P(x), s\rangle = \langle x, s\rangle - \langle P(x), s\rangle = \langle x, s\rangle - \langle x, s\rangle = 0.$$

That is, $x - P(x) \in S^\perp = \overline{\operatorname{span} S}^\perp$. By Theorem 6.8.7 this implies that P is the orthogonal projector onto $\overline{\operatorname{span} S}$. □

Proposition 6.23. *Let H be a Hilbert space and $S \subset H$ an orthonormal system. Then, the following are equivalent:*

1. *S is an orthonormal basis.*
2. *Suppose $x \in H$ and $x \perp S$. Then, $x = 0$.*
3. *$H = \overline{\operatorname{span} S}$.*
4. *$x = \sum_{s\in S} \langle x, s\rangle s \quad \forall x \in H$.*
5. *$\langle x, y\rangle = \sum_{s\in S} \langle x, s\rangle\langle s, y\rangle \quad \forall x, y \in H$.*
6. *$\|x\|^2 = \sum_{s\in S} |\langle x, s\rangle|^2 \quad \forall x \in H$.*

Proof. 1.⇒2.: If there exists $x \in S^\perp \setminus \{0\}$ then $S \cup \{x/\|x\|\}$ would be an orthonormal system strictly containing S, contradicting the maximality of S. 2.⇒3.: Note that $H = \{0\}^\perp = (S^\perp)^\perp = (\overline{\operatorname{span} S}^\perp)^\perp = \overline{\operatorname{span} S}$. 3.⇒4.: $\mathbf{1}(x) = P_{\overline{\operatorname{span} S}}(x) = \sum_{s \in S} \langle x, s\rangle s$ by Proposition 6.22. 4.⇒5.: Apply $\langle \cdot, y\rangle$. Since the inner product is continuous in the left argument, its application commutes with the limit taken in the sum. 5.⇒6.: Insert $y = x$. 6.⇒1.: Suppose S was not an orthonormal basis. Then there exists $y \in H \setminus \{0\}$ such that $y \in S^\perp$. But then $\|y\|^2 = \sum_{s \in S} |\langle y, s\rangle|^2 = 0$, a contradiction. □

Proposition 6.24. *Let H be a Hilbert space. Then, H admits an orthonormal basis.*

Proof. **<u>Exercise</u>.**Hint: Use Zorn's Lemma. □

Proposition 6.25. *Let H be a Hilbert space and $S \subset H$ an orthonormal basis of H. Then, S is countable iff H is separable.*

Proof. Suppose S is countable. Let $\mathbb{Q}S$ denote the set of linear combinations of elements of S with coefficients in $\mathbb{Q}$. Then, $\mathbb{Q}S$ is countable and also dense in H by using Proposition 6.23.3, showing that H is separable. Conversely, suppose that H is separable. Observe that $\|s - t\| = \sqrt{2}$ for $s, t \in S$ such that $s \neq t$. Thus, the open balls $B_{\sqrt{2}/2}(s)$ for different $s \in S$ are disjoint. Since H is separable there must be a countable subset of H with at least one element in each of these balls. In particular, S must be countable. □

In the following, we denote by $|S|$ the cardinality of a set S.

Proposition 6.26. *Let H be a Hilbert space and $S, T \subset H$ orthonormal basis of H. Then, $|S| = |T|$.*

Proof. If S or T is finite this is clear from linear algebra. Thus, suppose that $|S| \geq |\mathbb{N}|$ and $|T| \geq |\mathbb{N}|$. For $s \in S$ define $T_s := \{t \in T : \langle s, t\rangle \neq 0\}$. By Lemma 6.19, $|T_s| \leq |\mathbb{N}|$. Proposition 6.23.2 implies that $T \subseteq \bigcup_{s \in S} T_s$. Hence, $|T| \leq |S| \cdot |\mathbb{N}| = |S|$. Using the same argument with S and T interchanged yields $|S| \leq |T|$. Therefore, $|S| = |T|$. □

Proposition 6.27. *Let H_1 be a Hilbert space with orthonormal basis $S_1 \subset H_1$ and H_2 a Hilbert space with orthonormal basis $S_2 \subset H_2$. Then, H_1 is isomorphic to H_2 iff $|S_1| = |S_2|$.*

Proof. **<u>Exercise</u>.** □

Exercise 31. Let S be a set. Define $\ell^2(S)$ to be the set of maps $f : S \to \mathbb{K}$ such that $\sum_{s\in S} |f(s)|^2$ converges absolutely. (a) Show that $\ell^2(S)$ forms a Hilbert space with the inner product $\langle f, g\rangle := \sum_{s\in S} f(s)\overline{g(s)}$. (b) Let H be a Hilbert space with orthonormal basis $S \subset H$. Show that H is isomorphic to $\ell^2(S)$ as a Hilbert space.

Example 6.28. Recall the Banach spaces of Example 3.55, where X is a measurable space with measure μ. The space $\mathrm{L}^2(X, \mu, \mathbb{K})$ is a Hilbert space with inner product

$$\langle f, g\rangle := \int_X f\overline{g}.$$

Exercise 32. Let S^1 be the unit circle with the algebra of Borel sets and μ the Lebesgue measure on S^1. Parametrize S^1 with an angle $\phi \in [0, 2\pi)$ in the standard way. Show that $\{\phi \mapsto e^{\mathrm{i}n\phi}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathrm{L}^2(S^1, \mu, \mathbb{C})$.

Exercise 33. Equip the closed interval $[-1, 1]$ with the algebra of Borel sets and the Lebesgue measure μ. Consider the set of monomials $\{x^n\}_{n\in\mathbb{N}}$ as functions $[-1, 1] \to \mathbb{C}$ in $\mathrm{L}^2([-1, 1], \mu, \mathbb{C})$. (a) Show that the set $\{x^n\}_{n\in\mathbb{N}}$ is linearly independent and dense. (b) Suppose an orthonormal basis $\{s_n\}_{n\in\mathbb{N}}$ of functions $s_n \in \mathrm{L}^2([-1, 1], \mu, \mathbb{C})$ is constructed using the algorithm of Gram-Schmidt (Proposition 6.17) applied to $\{x^n\}_{n\in\mathbb{N}}$. Define $p_n := \sqrt{2/(2n + 1)}s_n$. Show that

$$(n + 1)p_{n+1}(x) = (2n + 1)xp_n(x) - np_{n-1}(x) \quad \forall x \in [-1, 1], \forall n \in \mathbb{N} \setminus \{1\}.$$

6.4 Operators on Hilbert Spaces

Definition 6.29. Let H_1, H_2 be Hilbert spaces and $\Phi_i : H_i \to H_i^*$ the associated anti-linear bijective isometries from Theorem 6.6. Let $A \in \mathrm{CL}(H_1, H_2)$ and $A^* : H_2^* \to H_1^*$ its adjoint according to Definition 4.27. We say that $A^\star \in \mathrm{CL}(H_2, H_1)$ given by $A^\star := \Phi_1^{-1} \circ A^* \circ \Phi_2$ is the *adjoint operator of A in the sense of Hilbert spaces.*

In the following of this section, *adjoint* will always refer to the adjoint in the sense of Hilbert spaces.

Proposition 6.30. *Let H_1, H_2 be Hilbert spaces and $A \in \mathrm{CL}(H_1, H_2)$. Then, $A^\star$ is the adjoint of A iff*

$$\langle Ax, y\rangle_{H_2} = \langle x, A^\star y\rangle_{H_1} \quad \forall x \in H_1, y \in H_2.$$

Proof. **Exercise.** □

In the following, we will omit subscripts indicating to which Hilbert space a given inner product belongs as long as no confusion can arise.

Proposition 6.31. *Let H_1, H_2, H_3 be Hilbert spaces, $A, B \in \mathrm{CL}(H_1, H_2)$, $C \in \mathrm{CL}(H_2, H_3)$, $\lambda \in \mathbb{K}$.*

1. $(A + B)^\star = A^\star + B^\star$.
2. $(\lambda A)^\star = \overline{\lambda} A^\star$.
3. $(C \circ A)^\star = A^\star \circ C^\star$.
4. $(A^\star)^\star = A$.
5. $\|A^\star\| = \|A\|$.
6. $\|A \circ A^\star\| = \|A^\star \circ A\| = \|A\|^2$.
7. $\ker A = (A^\star(H_2))^\perp$ *and* $\ker A^\star = (A(H_1))^\perp$.

Proof. **Exercise.** □

Definition 6.32. Let H_1, H_2 be Hilbert spaces and $A \in \mathrm{CL}(H_1, H_2)$. Then, A is called *unitary* iff A is an isometric isomorphism.

Remark 6.33. It is clear that $A \in \mathrm{CL}(H_1, H_2)$ is unitary iff

$$\langle Ax, Ay \rangle = \langle x, y \rangle \quad \forall x, y \in H_1.$$

Equivalently, $A^\star \circ A = \mathbf{1}_{H_1}$ or $A \circ A^\star = \mathbf{1}_{H_2}$.

Definition 6.34. Let H be a Hilbert space and $A \in \mathrm{CL}(H, H)$. A is called *self-adjoint* iff $A = A^\star$. A is called *normal* iff $A^\star \circ A = A \circ A^\star$.

Proposition 6.35. *Let H be a Hilbert space and $A \in \mathrm{CL}(H, H)$ self-adjoint. Then,*

$$\|A\| = \sup_{\|x\| \le 1} |\langle Ax, x \rangle|.$$

Proof. Set $M := \sup_{\|x\| \le 1} |\langle Ax, x \rangle|$. Since $|\langle Ax, x \rangle| \le \|Ax\|\|x\| \le \|A\|\|x\|^2$, it is clear that $\|A\| \ge M$. We proceed to show that $\|A\| \le M$. Given $x, y \in H$ arbitrary we have

$$\begin{aligned}\langle A(x+y), x+y \rangle - \langle A(x-y), x-y \rangle &= 2\langle Ax, y \rangle + 2\langle Ay, x \rangle \\ &= 2\langle Ax, y \rangle + 2\langle y, Ax \rangle = 4\Re\langle Ax, y \rangle.\end{aligned}$$

Thus,

$$4\Re\langle Ax, y\rangle \le |\langle A(x+y), x+y\rangle| + |\langle A(x-y), x-y\rangle| \\ \le M(\|x+y\|^2 + \|x-y\|^2) = 2M(\|x\|^2 + \|y\|^2).$$

The validity of this for all $x, y \in H$ in turn implies

$$\Re\langle Ax, y\rangle \le M\|x\|\|y\| \quad \forall x, y \in H.$$

Replacing x with λx for a suitable $\lambda \in \mathbb{K}$ with $|\lambda| = 1$ yields

$$|\langle Ax, y\rangle| \le M\|x\|\|y\| \quad \forall x, y \in H.$$

Inserting now $y = Ax$ we can infer

$$\|Ax\| \le M\|x\|\forall x \in H,$$

and hence $\|A\| \le M$, concluding the proof. □

Proposition 6.36. *Let H be a complex Hilbert space and $A \in \mathrm{CL}(H, H)$. Then, the following are equivalent:*

1. *A is self-adjoint.*
2. *$\langle Ax, x\rangle \in \mathbb{R}$ for all $x \in H$.*

Proof. 1.⇒2.: For all $x \in H$ we have $\langle Ax, x\rangle = \langle x, Ax\rangle = \overline{\langle Ax, x\rangle}$. 2.⇒1.: Let $x, y \in H$ and $\lambda \in \mathbb{C}$. Then,

$$\langle A(x+\lambda y), x+\lambda y\rangle = \langle Ax, x\rangle + \overline{\lambda}\langle Ax, y\rangle + \lambda\langle Ay, x\rangle + |\lambda|^2\langle Ay, y\rangle.$$

By assumption, the left-hand side as well as the first and the last term on the right-hand side are real. Thus, we may equating the right hand side with its complex conjugate yielding,

$$\overline{\lambda}\langle Ax, y\rangle + \lambda\langle Ay, x\rangle = \lambda\langle y, Ax\rangle + \overline{\lambda}\langle x, Ay\rangle.$$

Since $\lambda \in \mathbb{C}$ is arbitrary, the terms proportional to λ and those proportional to $\overline{\lambda}$ have to be equal separately, showing that A must be self-adjoint. □

Corollary 6.37. *Let H be a complex Hilbert space and $A \in \mathrm{CL}(H, H)$ such that $\langle Ax, x\rangle = 0$ for all $x \in H$. Then, $A = 0$.*

Proof. By Proposition 6.36, A is self-adjoint. Then, by Proposition 6.35, $\|A\| = 0$. □

Exercise 34. Give a counter example to the above statement for the case of a real Hilbert space.

Proposition 6.38. *Let H be a Hilbert space and $A \in \mathrm{CL}(H,H)$ normal. Then,*

$$\|Ax\| = \|A^\star x\| \quad \forall x \in H.$$

Proof. For all $x \in H$ we have,

$$0 = \langle (A^\star \circ A - A \circ A^\star)x, x\rangle = \langle Ax, Ax\rangle - \langle A^\star x, A^\star x\rangle = \|Ax\|^2 - \|A^\star x\|^2.$$

□

Proposition 6.39. *Let H be a Hilbert space and $A \in \mathrm{CL}(H,H)$ with $A \neq 0$ a projection operator, i.e., $A \circ A = A$. Then, the following are equivalent:*

1. *A is an orthogonal projector.*
2. *$\|A\| = 1$.*
3. *A is self-adjoint.*
4. *A is normal.*
5. *$\langle Ax, x\rangle \geq 0$ for all $x \in H$.*

Proof. 1.⇒2.: This follows from Theorem 6.8.5. 2.⇒1.: Let $x \in \ker A$, $y \in F := A(H)$ and $\lambda \in \mathbb{K}$. Then,

$$\|\lambda y\|^2 = \|A(x + \lambda y)\|^2 \leq \|x + \lambda y\|^2 = \|x\|^2 + 2\Re\langle x, \lambda y\rangle + \|\lambda y\|^2.$$

Since $\lambda \in \mathbb{K}$ is arbitrary we may conclude $\langle x, y\rangle = 0$. That is, $\ker A \subseteq F^\perp$. On the other hand set $\tilde{F} := (\mathbf{1} - A)(H)$ and note that $\tilde{F} \subseteq \ker A$. But since $\mathbf{1} = A + (\mathbf{1} - A)$ we must have $F + \tilde{F} = H$. Given $\tilde{F} \subseteq F^\perp$ this implies $\tilde{F} = F^\perp$ and hence $\ker A = F^\perp$. Observe also that F is closed since A is a projector and hence $F = \ker(\mathbf{1} - A)$. By Theorem 6.8, A is an orthogonal projector. 1.⇒3.: Using Theorem 6.8.2 and 6.8.7, observe for $x, y \in H$:

$$\langle Ax, y\rangle = \langle Ax, Ay - (Ay - y)\rangle = \langle Ax, Ay\rangle = \langle Ax - (Ax - x), Ay\rangle = \langle x, Ay\rangle.$$

3.⇒4.: Immediate. 4.⇒1.: Combining Proposition 6.38 with Proposition 6.31 we have $\ker A = \ker A^\star = (A(H))^\perp$. Note also that $A(H)$ is closed since A is a projector. Thus, by Theorem 6.8, A is an orthogonal projection. 3.⇒5.: For $x \in H$ observe

$$\langle Ax, x\rangle = \langle A \circ Ax, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2 \geq 0.$$

5.⇒1.: Let $x \in \ker A$ and $y \in F := A(H)$. Then,

$$0 \le \langle A(x+y), x+y \rangle = \langle y, x+y \rangle = \|y\|^2 + \langle y, x \rangle.$$

Since x can be scaled arbitrarily, we must have $\langle y, x \rangle = 0$. Thus, $\ker A \subseteq F^\perp$. As above we may conclude that A is an orthogonal projector. □

Exercise 35. Let X be a normed vector space and Y a separable Hilbert space. Show that $\mathrm{KL}(X, Y) = \overline{\mathrm{CL}_{\mathrm{fin}}}(X, Y)$. [Hint: Use Proposition 4.36 and show that the assumptions of Proposition 4.37 can be satisfied.]

Exercise 36. Let $w \in \mathrm{C}([0,1], \mathbb{R})$ and consider the map $\langle \cdot, \cdot \rangle_w : \mathrm{C}([0,1], \mathbb{C}) \times \mathrm{C}([0,1], \mathbb{C}) \to \mathbb{C}$ given by

$$\langle f, g \rangle_w := \int_0^1 f(x)\overline{g(x)} w(x) \mathrm{d}x.$$

1. Give necessary and sufficient conditions for $\langle \cdot, \cdot \rangle_w$ to be a scalar product.

2. When is the norm induced by $\langle \cdot, \cdot \rangle_w$ equivalent to the norm induced by the usual scalar product

$$\langle f, g \rangle := \int_0^1 f(x)\overline{g(x)} \mathrm{d}x?$$

Exercise 37. Let S be a set and $H \subseteq F(S, \mathbb{K})$ a subspace of the functions on S with values in $\mathbb{K}$. Suppose that an inner product is given on H that makes it into a Hilbert space. Let $K : S \times S \to \mathbb{K}$ and define $K_x : S \to \mathbb{K}$ by $K_x(y) := K(y, x)$. Then, K is called a *reproducing kernel* iff $K_x \in H$ for all $x \in S$ and $f(x) = \langle f, K_x \rangle$ for all $x \in S$ and $f \in H$. Show the following:

1. If a reproducing kernel exists, it is unique.

2. A reproducing kernel exists iff the topology of H is finer than the topology of pointwise convergence.

3. If K is a reproducing kernel, then $\mathrm{span}(\{K_x\}_{x \in S})$ is dense in H.

4. Let H be the two-dimensional subspace of $\mathrm{L}^2([0,1], \mathbb{K})$ consisting of functions of the form $x \mapsto ax + b$. Determine its reproducing kernel.